Stat 155 Lecture 1 Notes

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1 Combinatorial Games

1.1 Subtraction game and definitions

Consider a "subtraction game" with 2 players and 15 chips. Players alternate moves, with player 1 starting. At each move, the player can remove 1 or 2 chips. A player wins when they take the last chip (so the other player cannot move).

Let x be the number of chips remaining. Suppose you move next. Can you guarantee a win? Let's look at a few examples. If $x \in \{1, 2\}$, the player who moves can take the remaining chip(s) and win. If $x = 3$, the second player has the advantage; no matter what player 1 does, player 2 will be presented with 1 or 2 chips.

Write N as the set of positions where the next player to move can guarantee a win, provided they play optimally. Write P as the set of positions where the other player, the player that moved previously, can guarantee a win, provided that they play optimally. So $0, 3 \in P$, $1, 2 \in N$. In the case of our original game, $15 \in P$.

Definition 1.1. A *combinatorial game* is a game with two players (players 1 and 2) and a set X of positions. For each player, there is a set of legal moves between positions, $M_1, M_2 \subseteq X \times X$ (current position, next position). Players alternately choose moves, starting from some starting position x_0, and play continues until some player cannot move. The game has a winner or loser and follows normal or misere play.

Definition 1.2. In a combinatorial game, *normal play* means that the player who cannot move loses the game.

Definition 1.3. In a combinatorial game, *misere play* means that the player who cannot move wins the game.

Definition 1.4. An *impartial game* has the same set of legal moves for both players; i.e. $M_1 = M_2$. A *partisan game* has different sets of legal moves for the players.

Definition 1.5. A *terminal position* for a player is a position in which the player has no legal move to another position; i.e. x is terminal for player i if there is no $y \in X$ with $(x, y) \in M_i$.

Definition 1.6. A combinatorial grame is *progressively bounded* if, for every starting position $x_0 \in X$, there is a finite bound on the number of moves before the game ends.

Definition 1.7. A *strategy* for a player is a function that assigns a legal move to each non-terminal position. If X_{NT} is the set of non-terminal positions for player i, then $S_i : X_{NT} \to X$ is a strategy for player i if, for all $x \in X_{NT}$, $(x, S_i(x)) \in M_i$.

Definition 1.8. A *winning strategy* for a player from position x is a strategy that is guaranteed to a result in a win for that player.

Example 1.1. The subtraction game is an impartial combinatorial game. The positions are $X = \{0, 1, 2, \ldots, 15\}$, and the moves are $\{(x, y \in X \times X : y \in \{x-1, x-2\}\}$. The terminal position for both players is 0. The game is played using normal play. It is progressively bounded because from $x \in X$, there can be no more than x moves until the terminal position. A winning strategy for any starting position $x \in N$ is $S(x) = 3\lfloor x/3 \rfloor$.

1.2 Combinatorial games as graphs

Impartial combinatorial games can be thought of as directed graphs. Think of the positions as nodes and the moves as directed edges between the nodes Terminal positions are nodes without outgoing edges.

Example 1.2. What does the graph look like for the subtraction game? Every edge from a node in P leads into a node in N. There is also an edge from every node in N to a node in P. The winning strategy chooses one of these edges.

Acyclic graphs correspond to progressively bounded games. $B(x)$ is the maximum length along the graph from node x to a terminal position.

1.3 Existence of a winning strategy

Theorem 1.1. *In a progressively bounded, impartial combinatorial game, $X = N \cup P$. That is, from any initial position, one of the players has a winning strategy.*

Proof. By definition, $N, P \subseteq X$, so $N \cup P \subseteq X$. We now show that $X \subseteq N \cup P$. For each $x \in X$, we induct on $B(x)$. If $B(x) = 0$, then we are in a winning position for one of the two players, so $x \in N \cup P$. Now suppose that $x \in N \cup P$ holds when $B(x) \leq n$. If $B(x) = n + 1$, then every legal move leads to y with $B(y) \leq n$, so $y \in N \cup P$. Consider all the legal next positions y. Either

1. All of these y are in N, which implies $x \in P$, or

2. Some legal move leads to a $y \in P$, which implies $x \in N$. □

1.4 Chomp

Chomp is an impartial combinatorial game. Two players take turns picking squares from a rectangular chocolate bar and eat everything above and to the right of the square they pick (including the square itself); the squares removed are called the "chomp." The positions are the non-empty subsets of a chocolate bar that are left-closed and below-closed. The moves are $\{(x, y) \in X \times X : y = x - \text{ chomp}\}$. The terminal position is when only the bottom left square remains. The game follows normal play.

Chomp is progressively bounded because from $x \in X$ with $|x|$ blocks remaining, there can be no more than $|x| - 1$ moves until the terminal position.

Theorem 1.2. *In chomp, every non-terminal rectangle is in N.*

Proof. We use a "strategy stealing" argument. From a rectangle $r \in X$, there is a legal move $(r, r') \in M$ that we can always choose to skip; that is, for any move $(r', s) \in M$, we also have $(r, s) \in M$. There are two cases:

1. $r' \in P$, which implies $r \in N$.
2. $r' \in N$. In this case, there is an $s \in P$ with $(r', s) \in M$, But then we know that $(r, s) \in M$, also implying $r \in N$. □